Anomalous diffusion in a self-similar random advection field

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The problem on passive scalar advection in random (statistically homogeneous) self-similar media is solved. Even in superdiffusive mode the solution is found not to possess "heavy" power tails. Instead, they are exponential (not Gaussian). The derivation is conducted by means of the scaling analysis.

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I. INTRODUCTION

Last time inadequacy of the classical diffusion model to describe tracer transport in highly disordered media is not already doubted [1,2]. Fractional diffusion models based on equations with fractional derivatives belong to the most promising alternative approaches (see, for example, [3,4]). These imply two essential consequences. The first one is the anomalous temporal dependence of the particles plume size at long times ($R \propto t^{\gamma}$ with $\gamma \neq 1/2$). The second is the possibility of power (instead of Gaussian) decay of concentration at large ($r \gg R$) distances—the "heavy tails." The same is implied by various Lévy-flight-based models [5].

Depending on the form of asymptotic behavior, concentration at large distances from the source can differ by many orders. Therefore the issue of the actual tail structure may be extremely important, e.g., for the reliability assessment of radioactive waste disposal in geological formations.

As far as the standard fractional-diffusion approach is, in general, formally mathematical and physically insufficiently justified [6–8], and since its narrowness is continuously becoming better realized [8–14], the mentioned conclusions need to be verified on specific physical models. One of them is a model of random advection with long-range velocity correlations. Particularly, such a velocity field can describe fluid flow in a medium with random fractal properties. Previously the model was investigated in [15,16], where a number of fractional diffusion regularities were verified. However, the results were obtained there within the frame of simplifying assumptions and the issue of heavy tails remains open.

In the present paper the random advection model is studied on the basis of the scaling invariance concept without any simplifying assumptions. Special attention is paid to the analysis of asymptotic mode of concentration behavior at large distances (concentration tails).

II. PROBLEM FORMULATION

A basis of the random advection model is the equation for particles concentration $c(\boldsymbol{r},t)$:

$$\frac{\partial c}{\partial t} + \boldsymbol{\nabla}(\boldsymbol{v}c) = 0. \tag{1}$$

Here $\boldsymbol{v}=\boldsymbol{v}(\boldsymbol{r})$ is the advection velocity. The quantity $\boldsymbol{v}(\boldsymbol{r})$ is a random function of coordinates $\langle \boldsymbol{v}(\boldsymbol{r})\rangle = 0$, where$\langle \boldsymbol{v}(\boldsymbol{r})\rangle$ is the average value over an ensemble of realizations [17]. The velocity field satisfies incompressibility equation div $\boldsymbol{v}=0$.

We assume that the velocity correlations at large distances decrease according to power law and the n-point velocity correlation function defined by the equality $K^{(n)}_{i_1 i_2,\ldots,i_n}(\boldsymbol{r}_1,\boldsymbol{r}_2,\ldots,\boldsymbol{r}_n) = \langle \boldsymbol{v}_{i_1}(\boldsymbol{r}_1)\boldsymbol{v}_{i_2}(\boldsymbol{r}_2),\ldots,\boldsymbol{v}_{i_n}(\boldsymbol{r}_n)\rangle$ is a uniform function of the order $-nh$ at $|\boldsymbol{r}_i - \boldsymbol{r}_j| \gg a$ (for all pairs of $\boldsymbol{r}_i$, $\boldsymbol{r}_j$). Here $h>0$ and a is a short-range truncation radius. The medium is also supposed to be statistically homogeneous and isotropic. Therefore for the pair correlation function$K^{(2)}_{ij}(\boldsymbol{r}_1-\boldsymbol{r}_2) \equiv \langle \mathbf{v}_i(\boldsymbol{r}_1)\mathbf{v}_j(\boldsymbol{r}_2)\rangle$ we have

$$K^{(2)}_{ii}(\boldsymbol{r}_1-\boldsymbol{r}_2) \cong V^2\left(\frac{a}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}\right)^{2h} \quad \text{at } |\boldsymbol{r}_1-\boldsymbol{r}_2| \gg a, \tag{2}$$

where V^2 is the characteristic value of $K^{(2)}_{ij}(\boldsymbol{r})$ at $|\boldsymbol{r}| \lesssim a$.

III. MACROSCOPIC TRANSPORT EQUATION

Tracer concentration averaged over an ensemble of medium realizations, $\bar{c}(\boldsymbol{r},t) \equiv \langle c(\boldsymbol{r},t)\rangle$, satisfies the standard macroscopic equation expressing the property of particle number conservation:

$$\frac{\partial \bar{c}}{\partial t} + \text{div}\, \boldsymbol{q} = 0. \tag{3}$$

Here, $q(\boldsymbol{r},t)$ is the macroscopic flux density that meets an obvious requirement—to be zero in case of uniform concentration distribution. With taking causality principle and linearity of the problem into account, we have

$$q_i(\boldsymbol{r},t) = -\int_{-\infty}^{t} dt' \int d\boldsymbol{r}' f_{ij}(\boldsymbol{r}',t') \frac{\partial \bar{c}(\boldsymbol{r}-\boldsymbol{r}',t-t')}{\partial r_j}. \tag{4}$$

The response tensor function $f_{ij}(\boldsymbol{r},t)$ is determined by the advection velocity distribution and obeys the property of a positive definiteness:

$$f_{ij}(\boldsymbol{r},t)s_i s_j > 0, \tag{5}$$

where s is an arbitrary vector. Equation (5) follows from principle of entropy increase.

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For certainty, we consider the initial-condition problem, $\bar{c}(\boldsymbol{r},0)=c^{(0)}(\boldsymbol{r})$, with no source. Then the average concentration $\bar{c}(\boldsymbol{r},t)$ and the quantity $c^{(0)}(\boldsymbol{r}')$ are connected by the relation

$$\bar{c}(\boldsymbol{r},t) = \int dr' G(\boldsymbol{r}-\boldsymbol{r}',t)c^{(0)}(\boldsymbol{r}'). \tag{6}$$

The Fourier-Laplace transform of Green's function $G(r,t)$ according to Eqs. (3) and (4) is

$$G_{kp} = [p - M(k,p)]^{-1}, \tag{7}$$

where

$$M(k,p) = -k_i k_j \int_0^\infty dt e^{-pt} \int d^3\boldsymbol{r} e^{-i\boldsymbol{kr}} f_{ij}(\boldsymbol{r},t). \tag{8}$$

Hereafter $k \equiv |\boldsymbol{k}|$, $r \equiv |\boldsymbol{r}|$.

IV. SCALING ANALYSIS

Within the model under consideration [see Eq. (2)] there is no space scale to characterize the system behavior at $r \gg a$. This allows us to take advantage of the ideas of critical phenomena theory [18] and consider transport processes at distances $r \gg a$ to be scale-invariant. In other words, we shall consider the macroscopic transport equation to be invariant with respect to the transformation

$$\boldsymbol{r} \to s\boldsymbol{r}, \tag{9}$$

with all the quantities of Eqs. (3) and (4) transforming as

$$A \to s^{-\Delta_A} A, \tag{10}$$

where the exponent Δ_A is termed as scaling dimension of the quantity A.

The scaling dimensions of the velocity, concentration, and Green's function follow from Eq. (2) and the property of particle number conservation:

$$\Delta_v = h, \quad \Delta_c = \Delta_G = 3. \tag{11}$$

Equations (3) and (4) make it possible to establish a relation between the time and the flux density scaling dimensions:

$$\Delta_t = 2 - \Delta_q. \tag{12}$$

With Eqs. (11) and (12) taken into account, the identity $q = \langle\langle vc \rangle\rangle$ results in the expressions:

$$\Delta_q = h + 3, \tag{13}$$

$$\Delta_t = -(1+h). \tag{14}$$

Using Eqs. (4), (13), and (14) one can also easily obtain

$$\Delta_f = 2h + 3. \tag{15}$$

Note that the results of this section are correct only in the case when transport properties are determined by a long-range part of velocity correlations (see below).

V. CONCENTRATION BEHAVIOR

Scaling analysis gives good grounds to determine the concentration behavior. According to the results of the previous section the response function $f_{ij}(\boldsymbol{r},t)$ may be represented in the form

$$f_{ij}(\boldsymbol{r},t) = \frac{(Va^h)^2}{r^{2h+3}} \chi_{ij}(\boldsymbol{n},\xi), \quad \boldsymbol{n} = \frac{\boldsymbol{r}}{r}, \tag{16}$$

$$\xi = \frac{r}{(a^h V t)^{1/(1+h)}} \quad \text{at } r \gg a;$$

$$f_{ij}(\boldsymbol{r},t) \sim \frac{V^2}{a^3} \quad \text{at } r \lesssim a. \tag{17}$$

Here $\chi_{ij}(\boldsymbol{n},\xi)$ is a dimensionless tensor function.

Now we address the properties of the Green's function (and, therefore, of concentration behavior) depending on the value of h. Consider cases $h>1$, $h<1$, and $h=1$ separately.

A. $h>1$

From Eqs. (16) and (17) we see that the main area to contribute the integral over $\boldsymbol{r}$ and t in Eq. (8) corresponds to $r \lesssim a$, $t \lesssim a/V$. As soon as we are interested in large-scale concentration distribution ($r \gg a$, $t \gg a/V$), we may put $\exp(-ikr-pt) \cong 1$ in the integrand of Eq. (8). Thus we arrive at

$$M = -Dk^2, \quad D \sim Va. \tag{18}$$

Substituting this into Eq. (7) we get $G_{kp}=[p+Dk^2]^{-1}$, which is the Fourier-Laplace transform of Green's function for the equation $\partial\bar{c}/\partial t = D\Delta\bar{c}$. Therefore the tracer transport corresponds to the classical diffusion regime when $h>1$.

Note that classical scaling $\Delta_t=-2$ differs from the one given by Eq. (14). This is caused by the fact that tracer transport at $h>1$ is determined by short-range velocity distribution where the scale invariance does not take place.

B. $h<1$

As seen from Eqs. (16) and (17), contribution of large distances ($r \gg a$) into the integral of Eq. (8) is predominant over that of small ones ($r \lesssim a$) in this case. Substituting a long-range expression (16) into Eq. (8) we have:

$$M(k,p) = -k_i k_j \int_0^\infty dt e^{-pt} \int d^3\boldsymbol{r} e^{-i\boldsymbol{kr}} \frac{(Va^h)^2}{r^{2h+3}} \chi_{ij}(\boldsymbol{n},\xi). \tag{19}$$

From here an important relation follows for the mass operator:

$$M(k,p) = -p\varphi(\eta), \quad \eta = k^2 \left(\frac{p}{Va^h}\right)^{-2/(1+h)}. \tag{20}$$

Making use of this expression we can represent the Green's function

$$G(r,t)=\int_{b-i\infty}^{b+i\infty}\frac{dp}{2\pi i}\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{e^{i\boldsymbol{kr}+pt}}{p-M(k,p)},\quad \mathrm{Re}\, b>0 \tag{21}$$

[see Eq. (7)] in the form:

$$G(r,t)=(a^hVt)^{-3/(1+h)}g(\xi) \tag{22}$$

with $g(0)\neq 0,\infty$ and $g(\xi)\to 0$ as $\xi\to\infty$. We conclude from Eq. (22) that the particles plume size at long times has anomalous temporal dependence

$$R(t)\sim(a^hVt)^{1/(1+h)}\quad \text{at } h<1, \tag{23}$$

which corresponds to superdiffusion transport mode. This conclusion coincides with the result of [15,16].

Let us now turn to the Green function's (and, therefore, concentration's) behavior at asymptotically large distances, $\xi\gg 1$, when $r\gg R(t)$. Performing integration over the angles in Eq. (21) and taking into account the property $M(-k,p)=M(k,p)$ we get an expression

$$G(\boldsymbol{r},t)=\frac{1}{r}\int_{b-i\infty}^{b+i\infty}\frac{dp}{2\pi i}\int_{-\infty}^{\infty}\frac{kdk}{(2\pi)^2 i}\frac{e^{ikr+pt}}{p-M(k,p)}. \tag{24}$$

To derive a long-distance asymptotic of G function we need to know the analytical properties of integrand in Eq. (24). As will be seen below, the main contribution into the integral of Eq. (24) at $\xi\gg 1$ is given by the area $|\mathrm{Im}\, p|\ll \mathrm{Re}\, p$, $\mathrm{Re}\, p>0$. So, analyzing the analytical properties we will accept $\mathrm{Im}\, p=0, p>0$. Consider the behavior of the mass operator $M(k,p)$ in the two limiting cases: $p\neq 0, k\to 0$ and $k\neq 0, p=0$.

When $p\neq 0$, the convergence of the integral on t in Eq. (19) is ensured by the multiplier e^{-pt}. The dependence of $M(k,p)$ on the k variable in this case is determined by a decaying regime of the function$f_{ij}(\boldsymbol{r},t)$ at $r\to\infty$. One can easily see that$f_{ij}(\boldsymbol{r},t)$ decreases faster than any negative-exponent power in this limit. Indeed, the very fact of the existence of arbitrarily high-order velocity correlation function in our problem means that the velocity distribution is determined by a functional that decreases at great velocity magnitudes very steeply. So, in essence, the velocity advection field is virtually limited in magnitude. In turn, this means that the function$f_{ij}(\boldsymbol{r},t)$ decays at $r\to\infty$ so fast that all its power moments on coordinate exist. Therefore the point $k=0, p\neq 0$ is regular for the function $M(k,p)$ and we may write the following expression for the mass operator at $p\neq 0, |\eta|\ll 1$:

$$M(k,p)=-p\sum_{n=0}^{\infty}a_n\eta^n,\quad \eta=k^2\left(\frac{p}{Va^h}\right)^{-2/(1+h)}. \tag{25}$$

In the second limiting case, when $k\neq 0, p=0$ and $\mathrm{Im}\, k=0$, we have from Eq. (19)

$$M(k,0)\cong -AVa^hk^{1+h}, \tag{26}$$

where the constant $A>0$ due to Eq. (5).

The established properties of the mass operator allow us to consider $M(k,p)$ as a real, positive, and analytical function when k and p are real and finite. This means that the integrand in Eq. (24) has no singularity at the real axis of k and so a long-range asymptotic of the G function is not powerlike. Therefore there are no heavy (powerlike) concentration tails in the random advection model.

It follows from Eq. (24) that at real p the nearest to the real axis of k singularities of the $M(k,p)$ function resides on the imaginary k axis. According to Eqs. (19), (20), and (26) these singularities correspond to branch points at $k=\pm i\kappa_b$, in the vicinity of which the mass operator approximately equals

$$M(k,p)\cong -BVa^h\frac{k^2}{p^{(1-h)/(1+h)}}(\eta-\eta_b)^{-(1-h)},$$

$$\eta_b=-\kappa_b^2\left(\frac{p}{Va^h}\right)^{-2/(1+h)}<0,\quad B>0. \tag{27}$$

According to Eq. (19), the function $M(k,p)$ at the imaginary axis interval $(-i\kappa_b, i\kappa_b)$ is real and positive when $\mathrm{Im}\, p=0$. At the same time, as seen from Eq. (27), it tends to infinity at $k\to\pm i\kappa_b$. Therefore we come to the conclusion that the nearest to the real axis of k singularities of the integrand in Eq. (24) at $\mathrm{Im}\, p=0$ are the two poles $k=\pm i\kappa_0$ with $0<\kappa_0<\kappa_b$. According to Eq. (20) the quantity $\kappa_0\equiv\kappa_0(p)$ is determined by the equation

$$\varphi(\eta_0)+1=0,\quad \kappa_0(p)=\left(\frac{p}{Va^h}\right)^{1/(1+h)}\sqrt{|\eta_0|},\quad \eta_0<0. \tag{28}$$

Now, bearing in mind the limit $\xi\gg 1$ [$r\gg R(t)$] for the G function, after shifting upwards the integration contour on the k variable in Eq. (24) we come to the expression

$$G(r,t)\cong\frac{1}{4\pi i r\varphi'(\eta_0)}\int_{b-i\infty}^{b+i\infty}\frac{dp}{2\pi i p}\left(\frac{p}{Va^h}\right)^{2/(1+h)}\times\exp\{-\Gamma(p;r,t)\}, \tag{29}$$

where

$$\Gamma(p;r,t)=\kappa_0(p)r-pt. \tag{30}$$

An important area of p in the integral of Eq. (29) at $\xi\gg 1$ corresponds to inequality $|\Gamma|\gg 1$. This allows us to use the saddle-point technique to perform integration on p. As a result, we come to the final asymptotic expression for the Green's function in the long-distance limit:

$$G(r,t)\cong\frac{C}{(4\pi)^{3/2}}(Va^ht)^{-3/(1+h)}\varepsilon^{3(1-h)/2h}\exp(-h\varepsilon^{(1+h)/h}). \tag{31}$$

Here we use the following notations:

$$\varepsilon=\frac{\sqrt{|\eta_0|}}{1+h}\xi\equiv\frac{\sqrt{|\eta_0|}}{1+h}\frac{r}{(a^hVt)^{1/(1+h)}}, \tag{32}$$

$$C = \frac{1}{\varphi'(\eta_0)}\sqrt{\frac{2|\eta_0|}{h(1+h)}} \sim 1. \tag{33}$$

Therefore, according to Eq. (31), the concentration tail in random advection model at $h<1$ is of exponential kind [19].

Note that the saddle-point value of p [determined by $\partial\Gamma(p;r,t)/\partial p=0$] is real and positive; an effective range of integration in Eq. (29) corresponds to an inequality

$$\frac{|\mathrm{Im}\, p|}{|p|} \lesssim \xi^{-(1+h)/2h} \ll 1, \tag{34}$$

justifying the neglect of the imaginary part of p during the analysis of the analytical structure of the mass operator.

C. $h=1$

An attempt to consider this case as a limit $h\to 1$ from above using the response function representation of Eq. (16) meets a logarithmic divergence in Eq. (4). Hence we conclude that, in fact, the classical scaling $\xi \sim r/\sqrt{Dt}$ with $D \sim Va$ is modified by a weak coordinate dependence of a logarithmic kind:

$$\xi = r/\sqrt{D(r)t}. \tag{35}$$

This means that when $R(t) \gg R(0)$ at distances $r \lesssim R(t)$ the relation between the flux density $q(r,t)$ and concentration should effectively have the form:

$$\boldsymbol{q}(\boldsymbol{r},t) = -D(r)\,\boldsymbol{\nabla}\,\bar{c}(\boldsymbol{r},t) \tag{36}$$

with r counted from the center of initial concentration distribution. Substituting Eq. (35) into Eq. (16) and then the latter into Eq. (4) we get

$$\frac{dD(r)}{dr} \sim \frac{(Va)^2}{rD(r)}. \tag{37}$$

Here from it follows

$$D(r) = \tilde{D}\ln^{1/2}\left(\frac{r}{a}\right), \quad \tilde{D} \sim Va. \tag{38}$$

Substituting this relation into Eq. (35) we obtain an estimate for the particles plume size at long times:

$$R(t) \sim \sqrt{\tilde{D}t}\ln^{1/4}\left(\frac{\sqrt{\tilde{D}t}}{a}\right) \quad \text{at } h=1. \tag{39}$$

Turning to the long-distance concentration asymptotic we substitute Eqs. (35) and (38) into Eq. (19) to calculate the mass operator at the real axis of k with $\mathrm{Im}\, p=0$, $p>0$. This gives the relation

$$M(k,p) \cong -\tilde{D}k^2\ln^{1/2}\mu, \quad \mu = \min\left\{(ka)^{-1}, \sqrt{\frac{\tilde{D}}{a^2p}}\right\}. \tag{40}$$

Proceeding further as in the case $h<1$ we find that the nearest to the real axis of k singularities of the integrand in Eq. (24) are the two poles $k=\pm i\kappa_0$ with $\kappa_0 \cong \sqrt{p/\tilde{D}}\ln^{-1/4}(\sqrt{\tilde{D}/a^2p})$, which result in the following long-distance asymptotic for the G function at $\xi \gg 1$:

$$G(\boldsymbol{r},t) \cong \frac{1}{\{4\pi\tilde{D}t\ln^{1/2}(\tilde{D}t/ar)\}^{3/2}}\exp\left\{-\frac{r^2}{4\tilde{D}t\ln^{1/2}(\tilde{D}t/ar)}\right\}. \tag{41}$$

Therefore, a logarithmically modified Gaussian concentration tail takes place in the case $h=1$.

VI. CONCLUSION

In summary, a transport of passive scalar in a static random advection field with power-law velocity correlations was studied in the frame of scaling analysis. We found that depending on the steepness of correlations decay (characterized by the index h) various qualitatively different regimes of transport take place. When the decay is fast ($h>1$), the transport is determined by "short jumps" and a classical diffusion regime is realized. When $h<1$, the advective transport is correlated at large distances, and its main mechanism corresponds to Lévy walks leading to a superdiffusion regime. At $h=1$, logarithmic corrections to classical-diffusion regularities appear.

The main conclusion drawn up from the study concerns the concentration behavior at large distances (at the "tails"). It was shown that the concentration decay in the superdiffusion regime is of contracted exponential type and is even faster than the Gaussian one in classical diffusion. This is in contrast with space-fractional diffusion models where the tails are of power type.

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